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MARSHALL-OLKIN MOMENT EXPONENTIAL DISTRIBUTION

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ABSTRACT

Marshall and Olkin (1997) proposed a new method to establish more flexible family of distributions by adding a parameter to baseline distribution. In this article, Marshall-Olkin moment exponential (MOME) distribution is introduced. Various structural properties of MOME distribution including survival function, hazard rate function, ordinary moments, moments about mean, conditional moments, Renyi's entropy, generalized entropy and median expressions are derived. Maximum likelihood (ML) method is applied to obtain parameter estimates of the MOME distribution and a simulation study is conducted to check the convergence of ML estimators of the parameters of MOME distribution. Application to a real data set is carried out to illustrate the flexibility of the model.

KEYWORDS

Marshall-Olkin moment exponential (MOME) distribution; maximum likelihood estimators; moments

1. INTRODUCTION

Marshall and Olkin (1997) introduced a new family of distributions by adding a parameter to obtain new families of distributions which are more flexible and represent a wide range of behavior than the original distributions. Moment exponential distribution plays major role in the analyses of lifetime and survival data. Many researchers used the Marshall-Olkin method to propose new distributions and studied their properties and parameter estimation. Ghitany et al. (2005) showed Marshall-Olkin extended Weibull which can be obtained as a compound distribution from exponential distribution. Since 2005, the Marshall-Olkin extended distributions have been widely studied in statistics and numerous authors have developed various classes of these distributions such as Marshall-Olkin extended Pareto (Ghitany, 2005), Marshall-Olkin extended gamma (Ristic et al., 2007), Marshall-Olkin extended Lomax using censored data (Ghitany et al., 2007) and Marshall- Olkin extend uniform distribution (Jose & Krishna, 2011). Moreover, the reliability properties of the extended linear failure-rate distributions were studied by Ghitany and Kotz (2007). Jayakumar and Mathew (2008) proposed a method based on adding two parameters in to a family distribution and considered as generalization to the method suggested by Marshall and Olkin (1997). Gupta et al. (2010a) estimated the reliability from Marshall-Olkin extended Lomax distribution. Gupta et al. (2010b) studied the effect of the tilt parameter on the monotonicity of the failure rate and estimated the turning points of the failure rate of the extended Weibull distribution. Gui (2013) introduced Marshall-Olkin power lognormal distribution and studied its statistical properties of the new distribution. Cordeiro and Lemonte (2013) studied some mathematical properties of Marshall-Olkin extended Weibull distribution. Also, they determined the moments of the order statistics and discussed the estimation of the parameters using maximum likelihood method. The moment exponential (ME) (or length biased) distribution was proposed by Dara (2012) and discussed hazard and reversed hazard rate functions of ME distribution. They used the probability density function (pdf) of ME distribution as:

$$g(x;\beta) = \beta^2 x e^{-\beta x}, \qquad x, \beta > 0. \tag{1}$$

The aim of this paper is to define and study a new lifetime model called Marshall-Olkin moment exponential distribution. Its main feature is that one additional parameter is inserted in equation (1) through Marshall-Olkin's method to provide more flexibility for the generated model.

This article will be organized as follows: In section 2, we define Marshall-Olkin moment exponential distribution and provide some plots for its pdf and hazard rate function (hrf). Some of its structural properties are also derived in section 2. In section 3, the maximum likelihood (ML) estimates of the unknown model parameters are provided. Application to a real data set is performed in section 4. Finally, in section 5, we provide some concluding remarks.

2. THE MOME MODEL AND SOME OF ITS STRUCTURAL PROPERTIES

In this section we define a MOME distribution. Here, we also find its pdf, cdf, survival function, hazard rate function, moments about origin and about mean, conditional moments, Renyi's entropy, generalized entropy and median expressions. A simulation study is conducted to check the convergence of ML estimators of the parameters of MOME distributions.

i) Marshal Olkin's ME Probability density function

Marshal Olkin's (1997) defined probability density function of random variable X as

$$g\left(x\right) = \frac{\alpha f\left(x\right)}{\left(1-\bar{\alpha}\bar{F}\left(x\right)\right)^2}, \alpha > 0 \ -\infty < x < \infty \tag{2}$$

From (1) and (2) we can obtain the Marshal Olkin's ME distribution. The pdf of MOME distribution is

$$g\left(x\right)=\frac{1}{\beta^{2}}\frac{\alpha x e^{-\frac{x}{\beta}}}{\left(1-\bar{\alpha}\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{2}},\ \alpha,\beta,x>0 \qquad (3)$$

where $g\left(x\right)>0$ for $x>0$ $\int_{0}^{\infty} g\left(x\right)dx=1$, (3) is MOME model with base line distribution is an ME distribution (1).

Pdf's Graph of MOME distribution

By fixing α and changing β (and vice versa) different shapes are as under

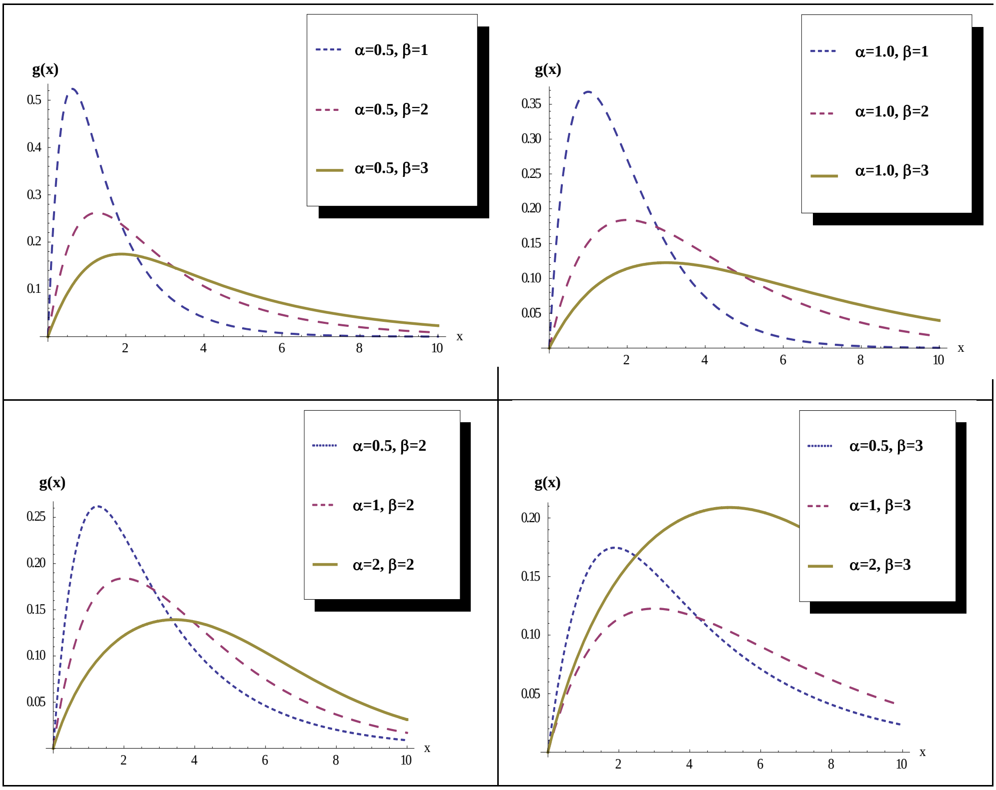


Fig. 1: The pdf of MOME distribution by fixing β and changing α (and vice versa)

ii) Cumulative distribution function of MOME distribution

$$G(x)=\int_0^x g(t)\,dt$$

$$G(x)=\int_0^x \frac{1}{\beta^2}\frac{\alpha t e^{-\frac{t}{\beta}}}{\left(1-\bar{\alpha}\left(1+\frac{t}{\beta}\right)e^{-\frac{t}{\beta}}\right)^2}dt$$

after some simplifications

$$G(x)=\frac{1-\left(1+\frac{x}{\beta}\right)e^{\frac{-x}{\beta}}}{1-(1-\alpha)\left(1+\frac{x}{\beta}\right)e^{\frac{-x}{\beta}}},\ x,\alpha,\beta>0 \tag{4}$$

where $G(0)=0,\quad G(\infty)=1$

iii) Survival function of MOME distribution

The function $S(x)=1-G(x)=P(X\geq x)$ is called survival function or reliability function.

$$S(x)=1-G(x)=\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)},\ \alpha>0 \text{ where } S(x)=\bar{G}(x)$$

$$S(x)=\frac{\alpha(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}}{1-\bar{\alpha}(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}},\ x,\alpha,\beta>0 \tag{5}$$

is survival function of MOME distribution.

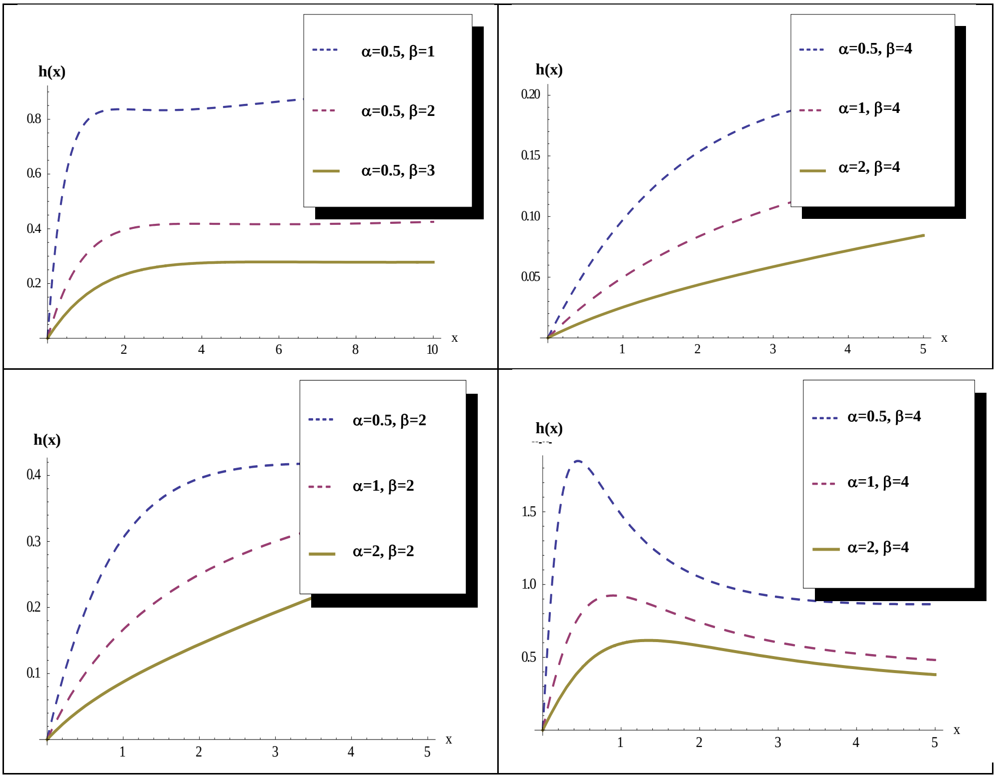
iv) Hazard rate function

The hazard rate function is defined as

$$h(x)=\frac{g(x)}{\bar{G}(x)}$$

$$h(x)=\frac{x}{\beta^2\left(1-(1-\alpha)\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)\left(1+\frac{x}{\beta}\right)},\ x,\alpha,\beta>0\ \ x>0 \tag{6}$$

Graph of hazard rate function of MOME for various values of α and β are



v) Reverse hazard rate function
Reverse hazard rate function is defined as

$$r(x)=\frac{g(x)}{G(x)}$$

$$r\left(x\right)=\frac{1}{\beta^2}\frac{\alpha x e^{-\frac{x}{\beta}}}{\left(1-\bar{\alpha}\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)\left(1-\left(1+\frac{x}{\beta}\right)e^{\frac{-x}{\beta}}\right)}, x,\alpha,\beta>0 \qquad (7)$$

vi) Mills ratio
Mills (1926) introduced the a ratio called Mills ratio defined by the equation as

$$m(x)=\frac{S(x)}{g(x)}$$

For the proposed MOME distribution Mills ratio is

$$m(x)=\frac{\beta^2}{x}\;(1+\frac{x}{\beta})\left(1-\bar{\alpha}\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right),\alpha,\beta,x>0 \qquad (8)$$

vii) Mean residual function

Mean residual life function (MRLF) e(x), for a random variable X with E(X) < ∞, is given as $e(x)=E\left(X-x\middle|X>x\right)$. It computes the average lifetime remaining for an item, which has already survived up to time x. it is given as $e(x)=\frac{1}{S\left(x\right)}\int_x^{\infty}\bar{G}(t)dt$

$$e(x)=\frac{1}{S(x)}\int_x^{\infty}\frac{\alpha(1+\frac{t}{\beta})e^{-\frac{t}{\beta}}}{1-\bar{\alpha}(1+\frac{t}{\beta})e^{-\frac{t}{\beta}}}dt$$

put $\frac{t}{\beta}=u$

$$e(x)=\frac{\alpha\beta}{S\left(x\right)}\int_{\frac{x}{\beta}}^{\infty}\frac{(1+u)e^{-u}}{1-\bar{\alpha}(1+u)e^{-u}}du$$

$$e(x)=\frac{\alpha\beta}{S\left(x\right)}\left(\int_{\frac{x}{\beta}}^{\infty}\frac{e^{-u}}{1-\bar{\alpha}(1+u)e^{-u}}du-\frac{1}{\bar{\alpha}}\ln\left(1-\bar{\alpha}(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}\right)\right)$$

By using following formula

$$\left(1-z\right)^{-k}=\sum_0^{\infty}\frac{\Gamma\left(k+j\right)}{\Gamma\left(k\right)\left(j\right)!}z^j \text{ and } \Gamma\left(s,x\right)=\int_x^{\infty}t^{s-1}e^{-t}\ dt$$

therefore, we have

$$\int_{\frac{x}{\beta}}^{\infty}\frac{e^{-u}}{1-\bar{\alpha}(1+u)e^{-u}}du=\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\frac{\binom{j}{k}}{\left(j+1\right)^{k+1}}\Gamma\left(k+1,\frac{\left(j+1\right)x}{\beta}\right),\alpha,\beta>0$$

finally,

$$e(x)=\frac{\alpha\beta}{S\left(x\right)}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\frac{\binom{j}{k}}{\left(j+1\right)^{k+1}}\Gamma\left(k+1,\frac{\left(j+1\right)x}{\beta}\right)-\frac{1}{\bar{\alpha}}\ln\left(1-\bar{\alpha}(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}\right)\right)$$

and hence

$$e(x)=\frac{\beta}{(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}}\left(1-\bar{\alpha}(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}\right)\times$$

$$\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^{j}\frac{\binom{j}{k}}{\left(j+1\right)^{k+1}}\Gamma\left(k+1,\frac{\left(j+1\right)x}{\beta}\right)-\frac{1}{\bar{\alpha}}\ln\left(1-\bar{\alpha}(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}\right)\right),\quad x,\alpha,\beta>0 \tag{9}$$

where

$$\Gamma\left(s,x\right)=\int_{x}^{\infty}t^{s-1}e^{-t}\,dt$$

viii) Vitality function

The vitality function v(x), of a random variable X with an absolutely continuous distribution function F(x) is given as $v(x)=E\left[X\,\middle|X>x\right]=\frac{1}{\bar{F}(x)}\int_{x}^{\infty}t\,dF(t)$

$$v(x)=\frac{1}{S(x)}\int_{x}^{\infty}t\,f(t)dt$$

$$=\frac{\alpha}{\beta^{2}S(x)}\int_{x}^{\infty}t\frac{te^{-\frac{t}{\beta}}}{\left(1-\bar{\alpha}\left(1+\frac{t}{\beta}\right)e^{-\frac{t}{\beta}}\right)^{2}}dt$$

Put $\frac{t}{\beta}=u$ $v(x)=\frac{\alpha\beta}{S\left(x\right)}\int_{\frac{x}{\beta}}^{\infty}\frac{u^{2}e^{-u}}{\left(1-\bar{\alpha}(1+u)e^{-u}\right)^{2}}du$,

after some simplifications, we have

$$v(x)=\frac{\alpha\beta}{S\left(x\right)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^{j}\binom{j}{k}\left(j+1\right)\int_{\frac{x}{\beta}}^{\infty}u^{k+2}e^{-u\left(j+1\right)}du\,,$$

$$v(x)=\frac{\beta\left(1-\bar{\alpha}(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}\right)}{(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^{j}\binom{j}{k}\frac{\Gamma\left(k+3,\frac{x\left(j+1\right)}{\beta}\right)}{\left(j+1\right)^{k+2}}\,, \tag{10}$$

where $\Gamma\left(s,x\right)=\int_{x}^{\infty}t^{s-1}e^{-t}\,dt$.

ix) Moments of MOME distribution

Moments about zero (Raw moments) for MOME are

$$E\left(X^r\right)=\int_0^\infty x^r g(x)dx$$

$$=\frac{1}{\beta^2}\int_0^\infty \frac{x^r \alpha x e^{-\frac{x}{\beta}}}{\left(1-\bar{\alpha}\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^2}dx$$

$$=\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}(j+1)\int_0^\infty \frac{x^{r+k+1}}{\beta^{k+2}}e^{-\frac{x}{\beta}(j+1)}dx$$

put $\frac{j+1}{\beta}x=t$

$$=\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}\frac{\beta^r}{(j+1)^{k+r+1}}\int_0^\infty t^{k+r+1}e^{-t}dt$$

$$=\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}\frac{\beta^r}{(j+1)^{k+r+1}}\Gamma(k+r+2)$$

$$E\left(X^r\right)=\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}\frac{\beta^r}{(j+1)^{k+r+1}}\Gamma(k+r+2)\,. \tag{11}$$

The proof is simply by comparing both methods.

$$\mu_1' =\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}(j+1)\frac{\beta}{(j+1)^{k+2}}\Gamma(k+3) \tag{12}$$

$$\mu_2' =\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}(j+1)\frac{\beta^2}{(j+1)^{k+3}}\Gamma(k+4) \tag{13}$$

$$\mu_3' =\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}(j+1)\frac{\beta^3}{(j+1)^{k+4}}\Gamma(k+5) \tag{14}$$

$$\mu_4' =\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}(j+1)\frac{\beta^4}{(j+1)^{5}}\Gamma(k+6) \tag{15}$$

Table 1
Values of the Moments about Origin for Different Values of α and β

α,β	μ_1'	μ_2'	μ_3'	μ_4'
0.5,1	1.52	3.82	13.72	129.1
0.5 ,2	3.05	15.27	109.74	1032.62
0.5,3	4.6	34.4	370.4	5227.65
0.9,1	1.92	5.62	22.1	109.4
0.9,2	3.844	22.46	176.75	1750.32
0.9,3	5.8	50.54	596.54	8861
1.9,1	2.5	8.14	38.98	207.65
1.9,2	5.03	35.26	311.87	3322.4
1.9,3	7.55	79.33	1052.36	16819.6

Values of the each moment about origin increases for increasing values of α and β.
Moments about mean (central moments)
The rth moment about mean

$$E\left(X-\mu\right)^{r}=\int_{0}^{\infty}\left(X-\mu\right)^{r}g(x)dx$$

Using the series

$$\left(a-b\right)^{k}=\sum_{0}^{k}(-1)^{k}\binom{k}{l}b^{k}\left(a\right)^{k-l}$$

$$=\sum_{l=0}^{r}(-1)^{l}\mu^{l}\binom{r}{l}\int_{0}^{\infty}X^{r-l}g(x)dx$$

from (11) we can easily deduce the following

$$\int_{0}^{\infty}X^{r-l}g(x)dx=\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^{j}\binom{j}{k}\frac{\beta^{r-l}}{\left(j+1\right)^{k+r-l+1}}\Gamma\left(k+r-l+2\right)$$

$$E\left(X-\mu\right)^{r}=\alpha\sum_{l=0}^{r}(-1)^{l}\binom{r}{l}(\mu)^{l}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^{j}\binom{j}{k}\frac{\beta^{r-l}}{\left(j+1\right)^{k+r-l+1}}\Gamma\left(k+r-l+2\right) \quad (14)$$

Table 2
Values of the Moments about Mean for Different Values of α and β

α,β	μ_2	μ_3	μ_4
0.5,1	1.51	3.32	82.62
0.5,2	5.96	26.76	286.48
0.9,1	1.94	3.99	23.23
0.9,2	7.72	30.98	351.12
1.9,1	29.01	9.18	5.912
1.9,2	54.03	34.3	479.8

Table 3
Moment Ratios for Various Values of α and β

α,β	β_1	β_2
0.5,1	1.78	36.2
0.5,2	1.84	8.1
0.9,1	1.47	6.2
0.9,2	1.43	5.89
1.9,1	0.06	2.96
1.9,2	0.086	0.16

For $\alpha = 0.9$ and $\beta=2$ the MOME distribution is approximately symmetric and normal.

x) Conditional moments

The rth conditional moments is defined as

$$E\left(X^r \mid X \le x\right) = \frac{1}{G(x)}\int_0^x t^r g(t)dt$$

$$= \frac{1}{G(x)}\int_0^x t^r \frac{\alpha f(t)}{\left(1-\bar{\alpha}\bar{F}(t)\right)^2} dt$$

$$= \frac{1}{G(x)}\sum_{j=0}^{\infty} \bar{\alpha}^j \frac{\Gamma(2+j)}{\Gamma(2)\, j!}\int_0^x t^r \left(\bar{F}(t)\right)^j f(t)\,dt$$

$$= \frac{1}{G(x)}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{\bar{\alpha}^j}{\beta^r (j+1)^{k+r+1}}\binom{j}{k}\int_0^{\frac{x(j+1)}{\beta}} u^{r+k+1}e^{-u}du$$

$$E\left(X^r \mid X \le x\right) = \frac{1}{G(x)}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{\bar{\alpha}^j}{\beta^r (j+1)^{k+r+1}}\binom{j}{k}\gamma\left(r+k+2, \frac{x(j+1)}{\beta}\right) \qquad (16)$$

$$\gamma(s,x) = \int_x^{\infty} t^{s-1}e^{-t}\, dt$$

xi) Inverse moments

The inverse moments are calculated as

$$E(X)^{-r} = \int_0^{\infty} x^{-r} g(x)dx$$

$$= \frac{1}{\beta^2}\int_0^\infty \frac{x^{-r}\alpha x e^{-\frac{x}{\beta}}}{\left(1-\bar{\alpha}\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^2}dx$$

$$= \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}(j+1)\int_0^\infty \frac{x^{k-r+1}}{\beta^{k+2}}e^{-\frac{x}{\beta}(j+1)}dx$$

$$E(X)^{-r} = \frac{\alpha}{\beta^r}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\frac{\binom{j}{k}}{(j+1)^{k-r+1}}\Gamma(k-r+2),\ \text{r-2 < k}, \quad (17)$$

xii) Median

Median of MOME distribution can be obtained by solving $G(m)=0.5$, where G(m) is the cdf of MOME distribution.

$$1-(1+\alpha)\left(1+\frac{m}{\beta}\right)e^{\frac{-m}{\beta}}=0 \quad (18)$$

Table 4
Values of the Median for Different Values of α and β

β	α					
	0.5	1	2	3	4	9
0.5	0.59	0.84	1.145	1.35	1.49	1.945
1	1.19	1.68	2.29	2.7	2.99	3.88
2	2.378	3.36	4.58	5.4	5.98	7.78
3	3.5	5.04	6.87	8.1	8.98	11.67
5	5.94	8.4	11.45	13.5	14.97	19.45

Values of the median increase for increasing α and β.

xiv) Entropy

Simply entropy means randomness. The idea of entropy in information theory was developed by Shannon (1948). It is a quantitative measure of uncertainty of information related to a random phenomenon. Like measure of dispersion, low entropy in a distribution indicates more concentration and more information as compared to high entropy. Entropy is very useful in reliability and survival analysis problems.

xv) Information Function (IF)

$$IF = E\left[\left(g\left(x\right)\right)^{S-1}\right] = \int_0^\infty \left(g\left(x\right)\right)^{S-1} g\left(x\right) dx$$

$$= \int_0^\infty \left(g\left(x\right)\right)^{S} dx = \int_0^\infty \left(1-\bar{\alpha}\bar{F}\left(x\right)\right)^{-2s} \alpha^s \left(f\left(x\right)\right)^s dx,$$

And after some simplification, we have

$$\mathrm{IF} = \alpha^s \sum_{j=0}^{\infty}\sum_{k=0}^{j} \bar{\alpha}^j \frac{\Gamma\left(2s+j\right)}{\Gamma\left(2s\right) j!}\binom{j}{k}\int_0^\infty \left(\frac{x}{\beta}\right)^k e^{-j\frac{x}{\beta}} \frac{x^s}{\beta^{2s}} e^{-\frac{x}{\beta}} dx$$

$$IF = \alpha^s \sum_{j=0}^{\infty}\sum_{k=0}^{j} \bar{\alpha}^j \frac{\Gamma\left(2s+j\right)}{\Gamma\left(2s\right) j!\beta^{s-1}}\binom{j}{k}\left(\frac{1}{j+s}\right)^{k+s+1} \Gamma\left(k+s+1\right) \qquad (20)$$

xvi) Re'nyi Entropy

Entropy has been used in various situations in science and engineering. The entropy of a random variable X is a measure of variation of the uncertainty. If X is a random variable which distributed as *MOME distribution*, then the Re'nyi entropy, for $\rho > 0$, and $\rho \neq 1$, is defined as

$$I_R(x) = (1-\rho)^{-1} \log_b\left(\int_0^\infty \left(f(x)\right)^\rho dx\right).$$

Let, $IP = \int_0^\infty \left(g(x)\right)^\rho dx$, then IP can be written as follows:

$$IP = \int_0^\infty \left(\alpha x\, e^{-\frac{x}{\beta}}\right)^\rho \left(1-\bar{\alpha}\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{-2\rho} dx.$$

$$= \alpha^\rho \sum_{j=0}^{\infty}\sum_{k=0}^{j} \bar{\alpha}^j \frac{\left(2\rho+j\right)!\binom{j}{k}\beta^{\rho+1}\Gamma\left(k+\rho+1\right)}{\left(2\rho\right)!\, j!\left(j+\rho\right)^{k+r+1}}$$

$$I_R(x) = (1-\rho)^{-1} \log_b\left(\alpha^\rho \sum_{j=0}^{\infty}\sum_{k=0}^{j} \bar{\alpha}^j \frac{\left(2\rho+j\right)!\binom{j}{k}\beta^{\rho+1}\Gamma\left(k+\rho+1\right)}{\left(2\rho\right)!\, j!\left(j+\rho\right)^{k+r+1}}\right).$$

xvii) Generalized Entropy

Generalized entropy is defined as

$$I(\lambda)=\frac{v_\lambda \mu^{-\lambda}-1}{\lambda\left(\lambda-1\right)}$$

where

$$v_\lambda=\int_{-\infty}^{\infty} x^\lambda f\left(x\right)dx \text{ and } \mu = mean$$

and for MOME distribution

$$v_\lambda=\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}\frac{\beta^\lambda}{\left(j+1\right)^{k+\lambda+1}}\Gamma\left(k+\lambda+2\right)$$

and

$$\mu=\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}\frac{\beta^\lambda}{\left(j+1\right)^{k+2}}\Gamma\left(k+3\right)$$

$$I(\lambda)=\frac{1}{\lambda\left(\lambda-1\right)}\left(\left(\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}\frac{\beta^\lambda}{\left(j+1\right)^{k+\lambda+1}}\Gamma\left(k+\lambda+2\right)\right)\left(\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}\frac{\beta^\lambda}{\left(j+1\right)^{k+2}}\Gamma\left(k+3\right)\right)^{-\lambda}-1\right)$$

$$I(\lambda)=\frac{\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}\frac{\beta^\lambda}{\left(j+1\right)^{k+\lambda+1}}\Gamma\left(k+\lambda+2\right)}{\lambda\left(\lambda-1\right)\left(\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\bar{\alpha}^j\binom{j}{k}\frac{\beta^\lambda}{\left(j+1\right)^{k+2}}\Gamma\left(k+3\right)\right)^\lambda}-\frac{1}{\lambda\left(\lambda-1\right)} \tag{23}$$

xix) Estimation of parameters

Suppose $X_1, X_2, ..., X_n$ is a random sample of size n drawn from equation (4.2) of MOME distribution

$$L(x;\alpha,\beta)=\prod_{i=1}^{n}\frac{1}{\beta^2}\frac{\alpha x e^{-\frac{x}{\beta}}}{\left(1-\bar{\alpha}\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^2}\alpha,\beta,x>0\,.$$

$$L(x;\alpha,\beta)=\frac{\alpha^n\prod_{i=1}^{n}x\prod_{i=1}^{n}e^{-\frac{x}{\beta}}}{\beta^{2n}\prod_{i=1}^{n}\left(1-\bar{\alpha}\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^2}\ \alpha,\beta,x>0$$

Its log-likelihood function is

$$l(x;\alpha,\beta)=n\ln\alpha+\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\left(\frac{x}{\beta}\right)\ln e-2n\ln\beta-2\sum_{i=1}^{n}\ln\left(1-\bar{\alpha}(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}\right)$$

The maximum likelihood estimators of unknown parameter α and β is that value of parameter which maximize the likelihood function, that can be obtained by solving the equations

$$\frac{\partial l(x;\alpha,\beta)}{\partial\alpha}=\frac{n}{\alpha}+\frac{n}{\alpha-1}-\sum_{i=1}^{n}\frac{(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}}{\left(1-\bar{\alpha}(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}\right)}$$

$$\frac{\partial l(x;\alpha,\beta)}{\partial\beta}=\sum_{i=1}^{n}(\frac{x}{\beta^2})-\frac{2n}{\beta}-2\sum_{i=1}^{n}\frac{\left(\bar{\alpha}(\frac{x}{\beta^2})e^{-\frac{x}{\beta}}-\bar{\alpha}(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}\left(\frac{x}{\beta^2}\right)\right)}{\left(1-\bar{\alpha}(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}\right)}$$

$$\frac{\partial l(x;\alpha,\beta)}{\partial\beta}=\sum_{i=1}^{n}(\frac{x}{\beta^2})-\frac{2n}{\beta}+2\sum_{i=1}^{n}\frac{\bar{\alpha}\frac{x^2}{\beta^3}e^{-\frac{x}{\beta}}}{\left(1-\bar{\alpha}(1+\frac{x}{\beta})e^{-\frac{x}{\beta}}\right)}$$

Putting $\frac{\partial l}{\partial\alpha}$ and $\frac{\partial l}{\partial\beta}$ equal to 0

$$\frac{n}{\hat{\alpha}}+\frac{n}{\hat{\alpha}-1}-\sum_{i=1}^{n}\frac{(1+\frac{x}{\hat{\beta}})e^{-\frac{x}{\hat{\beta}}}}{\left(1-\bar{\hat{\alpha}}(1+\frac{x}{\hat{\beta}})e^{-\frac{x}{\hat{\beta}}}\right)}=0 \qquad (24)$$

$$\sum_{i=1}^{n}\left(\frac{x}{\hat{\beta}^2}\right)-\frac{2n}{\hat{\beta}}+2\sum_{i=1}^{n}\frac{\bar{\hat{\alpha}}\frac{x^2}{\hat{\beta}^3}e^{-\frac{x}{\hat{\beta}}}}{\left(1-\bar{\alpha}(1+\frac{x}{\hat{\beta}})e^{-\frac{x}{\hat{\beta}}}\right)}=0 \tag{25}$$

(24) and (25) are not in closed form therefore for parameter estimation by maximum likelihood method we shall solve them numerically.

xx) Pdf of order statistics

The pdf of ith order statistics $X_{i,n}$ say $g_{i,n}(x)$ can be expressed by using equations

$$g_{i,n}(x)=\alpha n!\, f(x)\sum_{l=0}^{n-i}\frac{(-1)^l\left(F(x)\right)^{l+i}}{(i-1)!(n-i)!\left(1-\bar{\alpha}\bar{F}(x)\right)^{l+i-1}}$$

$$g_{i,n}(x)=\frac{\alpha n!}{\beta^2}xe^{-\frac{x}{\beta}}\sum_{j=0}^{n-i}\frac{(-1)^l\left(1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{l+i}}{(i-1)!(n-i)!\left(1-\bar{\alpha}\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{l+i-1}},\ \alpha,\beta,x>0$$

If $0<\alpha<1$ above function become

$$g_{i,n}(x)=f(x)\sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j}u_{j,l,k}\left(F(x)\right)^{j+l-k+i-1}$$

$$g_{i,n}(x)=\frac{x}{\beta^2}e^{-\frac{x}{\beta}}\sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j}u_{j,l,k}\left(1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right)^{j+l-k+i-1} \tag{26}$$

where

$$u_{j,l,k}=u_{j,l,k}(\alpha)=\frac{\alpha n!(-1)^l(1-\alpha)^j(-1)^{j-k}}{(i-1)!(n-i)!}\binom{j}{k}\binom{l+i+j}{j}$$

For $\alpha>1$ we write $1-\bar{\alpha}\bar{F}(x)=\alpha\left(1-\frac{(\alpha-1)F(x)}{\alpha}\right)$ then

$$g_{i,n}(x)=f(x)\sum_{j=0}^{\infty}\sum_{l=0}^{n-i}c_{j,l}\left(F(x)\right)^{j+l+i-1}$$

$$g_{i,n}(x) = \frac{x}{\beta^2} e^{-\frac{x}{\beta}} \sum_{j=0}^{\infty} \sum_{l=0}^{n-i} c_{j,l} \left(1 - \left(1 + \frac{x}{\beta}\right) e^{-\frac{x}{\beta}}\right)^{j+l+i-1} \quad (27)$$

where

$$c_{j,l} = c_{j,l}(\alpha) = \frac{n!(-1)^l (\alpha-1)^j}{\alpha^{l+j+i} (i-1)!(n-i)!} \binom{l+i+j}{j}$$

3. APPLICATION OF MOME DISTRIBUTION

To illustrate the performance of purposed MOME distribution we consider the data set obtained (see Aarset (1987)). It is a strength data measured in GPA, the single carbon fibers, and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge length 1 mm. The data are provided below:

2.247, 2.64, 2.908, 3.099, 3.126, 3.245, 3.328, 3.355, 3.383, 3.572, 3.581, 3.681, 3.726, 3.727, 3.728, 3.783, 3.785, 3.786, 3.896, 3.912, 3.964, 4.05, 4.063, 4.082, 4.111, 4.118, 4.141, 4.246, 4.251, 4.262, 4.326, 4.402, 4.457, 4.466, 4.519, 4.542, 4.555, 4.614, 4.632, 4.634, 4.636, 4.678, 4.698, 4.738, 4.832, 4.924, 5.043, 5.099, 5.134, 5.359, 5.473, 5.571, 5.684, 5.721, 5.998, 6.06

Before progressing further first we provide the histogram of the strength data in Figure below.

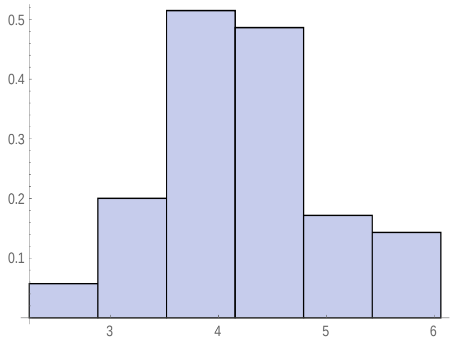


Fig 4.13: Histogram of Sample Data

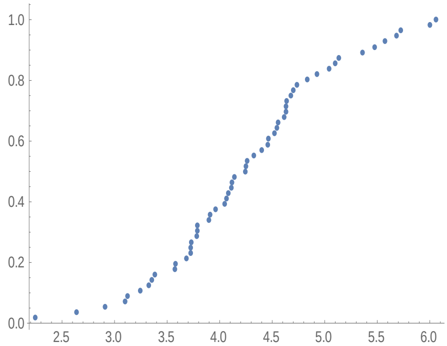


Fig 4.13: cdf of Sample Data

Note: From the above graph it is immediate that the data are unimodal.

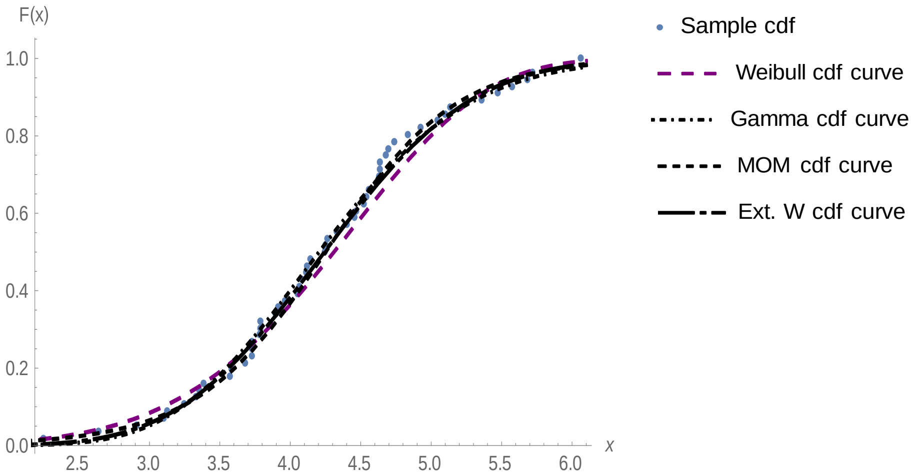


Figure 4.14: cdf Graphs of Weibull, Gamma, Extended Weibull and MOME Distributions

It is the comparison of cdfs of four mentioned distributions. It represents that graphs of all cdfs' have approximately same pattern.

Table 5
ML Estimates and Goodness of Fit Statistics for Strength Data

Parameters	**Weibull (λ,k)**	**Gamma (k,λ)**	**EW (a,b,c)**	**TEW (λ,β,k,α)**	**EXTW (a,b,c)**	**MOME (α,β)**
Estimates	6019.81	26.283	$1.0 * 10^{-7}$	0.23471	266.91145	2201.81
	5.7057	0.1620	0.000166	0.0001	0.00001	0.42073
	-	-	5.70572	0.0001	25.25964	-
	-	-	-	0.000001		-
	-	-	-	-		-
Log-likelihood	−68.9336	−68.379	−68.9336	−68.56634	−67.9508	−67.456
AIC	141.867	140.758	143.8672	142.33	141.9	138.912
BIC	145.918	144.809	149.94	148.409	147.976	142.963
Rank	5	3	5	4	2	1

EW: exponentiated Weibull distribution; EXTW: extended Weibull distribution; TEW: transmuted exponentiated Weibull.

From above table 5 we compare MOME model with Weibull, Gamma, EW, TEW and EXTW, we note that log-likelihood of MOME distribution is more than all distribution and its AIC and BIC both are less than AIC and BIC of other competitive distributions. So, we prove empirically that MOME distribution can be better model than all competitive models.

4. SIMULATION RESULTS

We simulate 10,000 samples of size n (30, 50, 100, 500) from MOME distribution with the specified values of parameters taking α=0.5, β=0.5 in table 5 and α=2, β=2. Equations (24) and (25) are the expressions for estimating the MLEs of the model parameters, which we use here for estimation of the distribution's parameters from the sample. A simulation is done by R Language. The following tables provide the information on estimated values, Bias and MSE.

Table 6
Simulation Results

Parameter	True value	Sample size	Estimated value	Bias	Variance	MSE
A	0.5	30	0.659892	0.1599	0.115816	0.141384
		50	0.613129	0.113129	0.062266	0.07506
		100	0.565157	0.065157	0.020461	0.024706
		500	0.5390341	0.03903	0.018358	0.01988
B	2	30	1.82815	0.17185	0.136606	0.166138
		50	2.08146	0.08146	0.12442	0.13106
		100	1.93215	0.06785	0.074591	0.07919
		500	1.9502	0.0498	0.033089	0.03557

Table 7
Simulation Results

Parameter	True value	Sample size	Estimated Value	Bias	Variance	MSE
α	2	30	1.84933	0.15067	0.579232	0.601933
		50	2.1357	0.1357	0.390657	0.40907
		100	2.05013	0.05013	0.336087	0.3386
		500	1.97363	0.02637	0.151871	0.152565
β	2	30	2.24336	0.24336	0.166985	0.22621
		50	1.88498	0.11502	0.065256	0.07848
		100	2.10112	0.1011	0.058283	0.06850
		500	2.00286	0.00286	0.018693	0.01870

5. CONCLUDING REMARKS

Adding parameters to a well-established distribution is used for obtaining more flexible new families of distributions. Marshall and Olkin (1997) used larger class of distribution functions by inserting an additional parameter in order to obtain a distribution function which contains the original one as the special case. In this paper, Marshall-Olkin moment exponential (MOME) distribution is introduced and properties of MOME distribution including survival function, hazard rate function, moments about origin, moments about mean, conditional moments, Renyi's entropy, generalized entropy and median expressions are derived. It is proved, empirically that, MOME distribution can be better model than all competitive models. A simulation study for the MOME model parameters are also included by taking different sample size. Bias and MSE of estimates of the parameters of MOME model are going to decrease for increasing n.

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